2005 IMO P4

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Solution to 2005 IMO P4

Problem: Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \ n \geq 1.$$

Solution: The answer is $\boxed{1}$, which clearly works. We will prove that there are no primes p relatively prime to all a_n. There are two cases.

$p = 2, 3$: Here, we can take $a_2 = 48$.

$p \neq 2, 3$: Here, we will prove that

$$a_{p-2} = 2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \equiv 0 \pmod{p}.$$

Note that

$$\begin{aligned} & 2^{p-2} + 3^{p-2} + 6^{p-2} - 1 \pmod{p} \\ \equiv & \frac{2^{p-1}}{2} + \frac{3^{p-1}}{3} + \frac{6^{p-1}}{6} - 1 \pmod{p} \\ \equiv & \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 \pmod{p} \\ \equiv & 0 \pmod{p}, \end{aligned}$$

which completes the problem.