2012 IMO P2

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Solution to 2012 IMO P2

Problem: Let a_2, a_3, $\ldots$, a_n be positive reals with product 1, where $n \geq 3$. Show that

$$(1+a_2)^2(1+a_3)^3\ldots(1+a_n)^n > n^n.$$

Solution: Note that

$$\begin{aligned} &1+a_k \\ &= \frac{1}{k-1}+\frac{1}{k-1}+\cdots+\frac{1}{k-1}+a_k \\ &\geq k\sqrt[k]{\frac{a_k}{(k-1)^{k-1}}} \end{aligned}$$

by AM-GM. Therefore,

$$(1+a_k)^k \geq k^k\left(\frac{a_k}{(k-1)^{k-1}}\right).$$

Multiplying these expressions together, we know that

$$(1+a_2)^2(1+a_3)^3\ldots(1+a_n)^n \geq n^n(a_2a_3\ldots a_n) = n^n.$$

Equality is impossible as that would imply

$$\frac{1}{k-1} = a_k$$

for every k, which is impossible.