2018 IMO P2

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Solution to 2018 IMO P2

Problem: Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n$ satisfying

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \ldots, n$, where indices are taken modulo n.

Solution: The answer is $n \equiv 0 \pmod 3$.

Proof that these n satisfy the condition. Take the sequence

$$(a_1, a_2, \ldots_{a_{n+2}}) = (-1, -1, 2, -1, -1, 2, -1, -1, 2, \ldots, -1, -1, -2, -1, -1),$$

which works as

$$-1 \cdot -1 + 1 = 2$$

and

$$-1 \cdot 2 + 1 = -1.$$

Proof that other n fail. Note that

$$a_k a_{k+1} a_{k+2} = (a_{k+2} - 1)(a_{k+2}) = a_{k+2}^2 - a_{k+2}.$$

It is also equal to

$$a_k(a_{k+3} - 1) = a_k a_{k+3} - a_k.$$

Therefore,

$$a_k a_{k+3} - a_k = a_{k+2}^2 - a_{k+2}.$$

Summing this for all $k = 1, 2, \ldots, n$ (indices $\pmod n$), we know that

$$\sum a_k a_{k+3} = \sum a_k^2,$$

or
$$\sum 2a_k a_{k+3} = 2\sum a_k^2.$$
To finish the problem, we know that
$$2\sum a_k^2 = \sum (a_k^2 + a_{k+3}^2),$$
so
$$\sum (a_k^2 + a_{k+3}^2) - 2\sum (a_k a_{k+3}) = 0.$$
This implies that
$$\sum (a_k - a_{k+3})^2 = 0,$$
or $a_k = a_{k+3}$. The sequence now has a period of 3, so $n \equiv 0 \pmod 3$. If n is not a multiple of 3, all the terms would be equal. This is impossible as
$$a_i^2 + 1 = a_i$$
does not have real roots.