2021 IMO P1

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Solution to 2021 IMO P1

Problem: Let $n \geq 100$ be an integer. Ivan writes the numbers n, $n+1$, $\ldots$, $2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution: We will three numbers a, b, and c so that the three sums

$$a + b = (2x - 1)^2 = 4x^2 - 4x + 1$$

$$b + c = (2x)^2 = 4x^2$$

$$c + a = (2x + 1)^2 = 4x^2 + 4x + 1$$

are perfect squares. This will finish the problem. Solving gives

$$a = 2x^2 + 1$$

$$b = 2x^2 - 4x$$

$$c = 2x^2 + 4x.$$

We need to prove that x exists, and we are done. We need

$$n \leq 2x^2 - 4x < 2x^2 + 1 < 2x^2 + 4x \leq 2n.$$

In particular, we want

$$n \leq 2x^2 - 4x$$

and

$$x^2 + 2x \leq n.$$

The intervals of n near 100 that work using this approach are

$$x = 8 \implies 80 \leq n \leq 96$$

$$x = 9 \implies 99 \leq n \leq 126$$

$$x = 10 \implies 120 \leq n \leq 160.$$

We need to prove that these intervals cover every number $n \geq 100$. In other words, we want to prove that the right endpoint of any interval $x \geq 9$ is larger than the left endpoint of the interval corresponding to $x + 1$. This is showing that

$$2x^2 - 4x \geq (x + 1)^2 + 2(x + 1) = x^2 + 4x + 3,$$

or

$$x^2 - 8x - 3 \geq 0.$$

This is clear for $x \geq 9$.