

2023 IMO P1

James Stewart

September 10, 2024

Solution to 2023 IMO P1

Problem: Determine all composite integers $n > 1$ that satisfy the following property: if d_1, d_2, \dots, d_k are all the positive divisors of n with $1 = d_1 < d_2 < \dots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k - 2$.

Solution: The answer is $n = p^k$ for any prime p and integer $k > 1$. These values of n clearly work.

Proof that no other values of n work. Suppose that n has at least two prime divisors. Call the smallest two prime factors a and b . The smallest factors of n are

$$1, a, a^2, \dots, a^x, b, \dots$$

for some value of x . Therefore,

$$b + a^x \equiv 0 \pmod{a^{x-1}},$$

and b is a multiple of a^{x-1} . This contradicts the assumption that a and b were primes.