2024 IMO P1

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Solution to 2024 IMO P1

Problem: Find all real numbers α such that, for every positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is divisible by n.

Solution: The answer is even integers.

There are two cases.

Case 1: α is an integer. We know that n is a factor of

$$\alpha + 2\alpha + \cdots + n\alpha = \alpha\frac{n(n-1)}{2},$$

so

$$\alpha(\frac{n-1}{2})$$

must be an integer. Since n can be an even number, α must be an even integer here.

Case 2: α is not an integer. We can translate α by multiples of 2 until $-1 < \alpha < 1$.

Note that

$$\lfloor \alpha + 2k \rfloor + \lfloor 2(\alpha + 2k) \rfloor + \cdots + \lfloor n(\alpha + 2k) \rfloor \equiv \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor \pmod{n}$$

if k is an integer.

Case 2.1: $0 < \alpha < 1$. Let k be the smallest positive integer satisfying $k\alpha > 1$. Setting $n = k$, we have

$$0 + 0 + \cdots + 0 + 0 + 1$$

is a multiple of n. This implies that $n = k = 1$, so $\alpha > 1$, a contradiction.

Case 2.2: $-1 < \alpha < 0$. Let k be the smallest positive integer satisfying $k\alpha < -1$. Setting $n = k$, we have

$$(-1) + (-1) + \cdots + (-1) + (-1) + (-2) = (-1)(n-1) + (-2) = -n - 1.$$

Since this is a multiple of n, we know that $n = k = 1$, so $\alpha < -1$, a contradiction.