

2014 JMO P1

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Solution to 2014 JMO P1

Problem: Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2 - 5a + 1}{b^2 - 5b + 10}, \frac{10b^2 - 5b + 1}{c^2 - 5c + 10}, \frac{10c^2 - 5c + 1}{a^2 - 5a + 10}\right) \leq abc.$$

Solution: We will show that

$$\frac{10x^2 - 5x + 1}{x^2 - 5x + 10} \leq x^3$$

for $x \geq 1$. Expanding, we want to prove that

$$10x^2 - 5x + 1 \leq x^5 - 5x^4 + 10x^3,$$

or

$$x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 = (x - 1)^5 \geq 0,$$

which is true. Therefore, the product of the three terms in the problem is $\leq a^3b^3c^3$. At least one of the terms is $\leq abc$.