2018 JMO P2

James Stewart

September 12, 2024

Solution to 2018 JMO P2

Problem: Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2)\geq a^2+b^2+c^2.$$

Solution: By WLOG and scaling, let $1=a\leq b\leq c$. We know that $1+b+c=4\sqrt[3]{bc}$ and want to prove that

$$2(b+c+bc)+4\geq 1+b^2+c^2.$$

If we let $x=\sqrt[3]{bc}$, we know that $b+c=4x-1$ and $bc=x^3$. We want to prove that

$$\begin{aligned}2(4x-1+bc)+3 &\geq (b+c)^2-2bc\\ &=(4x-1)^2-2bc\\ &=16x^2-8x+1-2bc\\ &=16x^2-8x+1-2x^3.\end{aligned}$$

Simplifying, we want to show that

$$2(4x-1+x^3)+3\geq 16x^2-8x+1-2x^3,$$

or

$$4x^3-16x^2+16x\geq 0.$$

This factors as

$$4x(x-2)^2\geq 0,$$

which is true.