

2011 USAMO P1

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Solution to 2011 USAMO P1

Problem: Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$.
Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

Solution: We know that $a^2 + b^2 + c^2 + ab + bc + ca \leq 2$. Note that

$$\begin{aligned} & \sum_{cyc} \frac{2ab+2}{(a+b)^2} \\ & \geq \sum_{cyc} \frac{2ab + (a^2 + b^2 + c^2 + ab + bc + ca)}{(a+b)^2} \\ & = \sum_{cyc} \frac{(a+b)^2 + (c+a)(c+b)}{(a+b)^2} \\ & = 3 + \sum_{cyc} \frac{(c+a)(c+b)}{(a+b)^2}. \end{aligned}$$

By AM-GM, this is

$$\geq 3 + 3\sqrt[3]{1} = 6.$$

This completes the problem.